

OPERATOR ALGEBRA OF TRANSVERSELY AFFINE FOLIATIONS

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ABSTRACT. We establish a geometric condition that determines when a type III von Neumann algebra arises from a foliation whose holonomy becomes affine with respect to a suitable transverse coordinate system. Under such an assumption the Godbillon-Vey class of the foliation becomes trivial in contrast to the case considered in Connes's famous theorem.

1. INTRODUCTION

To each foliated manifold $(M; F)$, one can associate a von Neumann algebra $W(M; F)$ of bounded operators labeled by the leaves of the foliation F . In [5] Connes showed that for any transversely oriented codimension 1 foliation $(M; F)$ that has nontrivial Godbillon-Vey class, the algebra $W(M; F)$ has a type III direct summand.

In this paper we investigate foliations whose holonomy maps are affine with respect to a suitable transverse coordinate system. When a foliation $(M; F)$ satisfies this condition, one can in particular deduce the vanishing of the Godbillon-Vey class. Under this assumption the transverse fundamental class of $(M; F)$ becomes invariant under the modular automorphism, and the cohomology class on the manifold corresponding to it is given by the "gradient" 1-form of the transverse density.

2. TRANSVERSE FUNDAMENTAL CLASS AND THE MODULAR GROUP OF TRANSVERSELY AFFINE FOLIATIONS

In the following we assume that all the manifolds and the foliations are smooth and oriented unless the contrary is explicitly stated. See [3] for the basic definitions and notations on foliation algebras.

2.1. Definition and elementary properties of the transverse fundamental class. Let M be a manifold of dimension n , F a foliation of dimension p and codimension q on M . In the following we assume that M and F are oriented, F is smooth and defines an ergodic equivalence relation on M unless the contrary is explicitly stated.

Definition 1. ([1]) The foliation F is said to be *transversely affine* when there exists a covering of M by foliation charts with respect to which the holonomy maps become affine transformations.

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Let G denote the holonomy groupoid $\text{Hol}(M; F)$ of $(M; F)$. One then has the smooth convolution algebra $\mathcal{A} = C_c^\infty(G, \Omega^{\frac{1}{2}})$ on G . Let H be a subbundle of TM complement to F . Then we have the transverse fundamental cocycle $[G]$, that is a cyclic q -cocycle on \mathcal{A} given by

$$(1) \quad (f^0, f^1, \dots, f^q) \mapsto \int \text{Tr}(f^0 d_H f^1 \cdots d_H f^q),$$

where the integral is taken in the transverse direction.

Remark 2. As noted in [6], for general foliations one needs to consider the algebra $M_2 \mathcal{A}$ instead of \mathcal{A} to overcome the failure of the holonomy invariance of vector fields tangent to H . However, for a transversely affine foliation there is a preferred choice of H , namely the subbundle associated to the linearizing transverse coordinate system. For this choice of H the expression 1 gives an actual cyclic q -cocycle over \mathcal{A} .

Remark 3. Let T be a transversal for F . By the ergodicity assumption on F , we may assume that it is contained in one foliation chart. Hence we may assume that there is a coordinate system (x_1, \dots, x_q) on the whole T with respect to which the restricted holonomy maps of F on T are affine.

For each transversal T , the restricted groupoid $G_T = \{\gamma \in G \mid r\gamma, s\gamma \in T\}$ is an étale groupoid over T . Its smooth convolution algebra $C_c^\infty(G_T)$ is strongly Morita equivalent to \mathcal{A} . The transverse fundamental class over G_T is given by the cyclic q -cocycle

$$\phi: (f^0, f^1, \dots, f^q) \mapsto \int_{\gamma_0 \cdots \gamma_q \in T} f_{\gamma_0}^0 df_{\gamma_1}^1 \cdots df_{\gamma_q}^q.$$

Suppose T is contained in a foliation chart as in the previous remark. Let $x = (x_1, \dots, x_q)$ be a transverse coordinate system on T . The volume form of this coordinate system determines a density on T . One derives a homomorphism

$$\delta: G_T \rightarrow \mathbb{R}, \delta(\gamma) = \frac{d\gamma}{dx}.$$

This determines a 1-parameter group σ_t on $C_c^\infty(G_T)$ by $\sigma_t(f)(\gamma) = \delta(\gamma)^{it} f(\gamma)$.

Remark 4. As noted in [3], that σ extends to the modular automorphism of the von Neumann algebra of $(M; F)$. The generator

$$Df = \lim_{t \rightarrow 0} \frac{\sigma_t f - f}{it}$$

of σ is given by the multiplication by the logarithm of δ as follows: $D.f(\gamma) = \log(\delta(\gamma))f(\gamma)$.

Proposition 5. *The transverse fundamental class ϕ on $C_c^\infty(G_T)$ is invariant under the modular automorphism σ_t associated to x .*

Proof. As in [5], the time derivative of $\sigma_t^* \phi$ is given by

$$\begin{aligned} & \sum_{1 \leq j \leq q} \int_{\gamma_0 \cdots \gamma_q \in T} i f^0(\gamma_0) f^j(\gamma_j) d \log \delta(\gamma_j) df^1(\gamma_1) \cdots d\check{f}^j(\gamma_j) \cdots df^q(\gamma_q) \\ & + \int_{\gamma_0 \cdots \gamma_q \in T} \sum_{0 \leq j \leq q} \log \delta(\gamma_j) f^0(\gamma_0) df^1(\gamma_j) \cdots df^q(\gamma_q), \end{aligned}$$

where $\check{d}f^j$ denotes the omission. By assumption $\delta(\gamma)$ is constant along transverse movement, hence $d \log \delta(\gamma_j) = 0$ for each term in the first part. For the second part, $\gamma_0 \cdots \gamma_q \in T$ implies $\delta(\gamma_0 \cdots \gamma_q) = 1$ and $\sum \log \delta(\gamma_j) = 0$, hence it is also zero. \square

Remark 6. When the codimension 1 foliation $(M; F)$ is transversely affine, the Godbillon-Vey class $GV \in H^3(M)$ that corresponds to the time derivative $\dot{\phi}$ of $t \mapsto \sigma_t^* \phi$ at $t = 0$, vanishes.

Example 7. Suppose $A \in SL_2 \mathbb{Z}$ is a hyperbolic matrix, i.e. it has two distinct positive real eigenvalues $0 < \lambda < 1 < \lambda^{-1}$. The linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserves the lattice \mathbb{Z}^2 and induces a transformation $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.

This action preserves the following foliation on \mathbb{T}^2 : let u, v be eigenvectors of A respectively corresponding to λ and λ^{-1} . They determine tangent vectors in $T_p \mathbb{T}^2$ at each point p of \mathbb{T}^2 . The subspaces $\mathbb{R}u \subset T_p \mathbb{T}^2$ for $p \in \mathbb{T}^2$ define a foliation F_u on \mathbb{T}^2 . This is a Kronecker foliation.

Let ρ be a holonomy invariant transverse measure for $(\mathbb{T}^2; F_u)$. For example ρ is represented by the usual angular measure on a transversal $T = S^1 \times \{\text{pt}\} \subset \mathbb{T}^2$. By $Av = \lambda^{-1}v$, A scales ρ by λ . Note that, on the other hand, A preserves the usual Haar measure and the fundamental class of \mathbb{T}^2 .

The von Neumann algebra $W(\mathbb{T}^2; F_u)$ of the foliation F_u is amenable of type II_∞ with a normal trace τ_ρ determined by ρ . The matrix A scales this normal trace by $A\tau = \lambda\tau$. Thus the crossed product algebra $W(\mathbb{T}^2; F_u)$ is of type III_λ ¹. This crossed product algebra is represented by the mapping torus foliation of the dynamical system $(\mathbb{T}^2; F_u) \curvearrowright_A \mathbb{Z}$: let \mathbb{T}_A^3 denote the mapping torus $\mathbb{T}^2 \times \mathbb{R}/(p, s) \sim (Ap, s + 1)$, F the foliation induced by $F_u \times T\mathbb{R}$ on $\mathbb{T}^2 \times \mathbb{R}$. The image of the transversal T for (\mathbb{T}^2, F_u) in $\mathbb{T}^2 \times \{0\} \subset \mathbb{T}_A^3$ is again a transversal to F and the holonomy maps are represented by local linear maps with respect to the standard angular coordinate on T .

2.2. The dual cocycle of the transverse fundamental class. Let $C_c^\infty(G_T) \rtimes_\sigma \mathbb{R}$ be the crossed product of $C_c^\infty(G_T)$ by σ . As a topological vector space it is the space of the $C_c^\infty(G_T)$ -valued Schwartz class functions on \mathbb{R} . The product is given by the twisted convolution $(f * g)_\lambda = \int_{\lambda=\mu+\nu} f_\mu \sigma_\mu(g_\nu)$. Since ϕ is invariant under σ , we obtain its dual q -cocycle [7]

$$(2) \quad \hat{\phi}(\xi^0, \dots, \xi^q) = \int_{\lambda_0 + \dots + \lambda_q = 0} \phi(\xi_{\lambda_0}^0, \sigma_{\lambda_0} \xi_{\lambda_1}^1, \dots, \sigma_{\lambda_0 + \dots + \lambda_{q-1}} \xi_{\lambda_q}^q)$$

on $\mathcal{A} \rtimes_\sigma \mathbb{R}$. We are going to construct $\hat{\sigma}$ -invariant measures on the flow of weights of $W(M; F)$ by means of this cyclic cocycle.

Definition 8. Let $G_T \rtimes_\delta \mathbb{R}$ denote the groupoid defined as follows:

- its object set is the direct product $G_T^0 \times \mathbb{R} = T \times \mathbb{R}$.
- its morphism set is again the direct product $G_T \times \mathbb{R}$, with the source map given by $s(\gamma, t) = (s\gamma, t)$ and the range map by $r(\gamma, t) = (r\gamma, t + \log \delta\gamma)$.

Via the Fourier transform

$$\hat{f}_\lambda(\gamma_x^y) = \int e^{it\lambda} f(\gamma_{(x,t)}^{(y,t+\log \delta\gamma)}) dt,$$

$\mathcal{S}(G_T \rtimes_\delta \mathbb{R})$ is isomorphic to $\mathcal{A} \rtimes_\sigma \mathbb{R}$.

¹This is also a consequence of [2].

Remark 9. The above groupoid G_T is given by the action of G_T on the total space of the (positive) density bundle $\wedge^q \tau_T^*$ over T . For each $x \in T$, the fiber $(\wedge^q \tau_T^*)_x$ over x is given by the linear maps of $\wedge^q T_x$ into R which respect the orientation. This is a closed manifold of the total space of the transverse density bundle $\wedge^q \tau^*$ over M , which is again a principal \mathbb{R}_+^\times bundle over M .

The tautological action of the holonomy groupoid G on $\wedge^q \tau^*$ defines a foliation F_u of codimension $q + 1$ on the $(n + 1)$ -dimensional manifold $\wedge^q \tau^*$. The closed submanifold $\wedge^q \tau_T^*$ is a transversal with respect to this foliation, while the algebra $C_c^\infty(G_T \rtimes \mathbb{R})$ is identified to the corresponding restricted groupoid algebra.

Lemma 10. *The q -cocycle $\hat{\phi}$ is equivalent to the following q -cocycle ψ*

$$\psi(f^0, \dots, f^q) = \int_{\mathbb{R}} dt \int_{\gamma_0 \dots \gamma_q \in T \times \{t\}} f_{\gamma_0}^0 df_{\gamma_1}^1 \wedge \dots \wedge df_{\gamma_q}^q$$

on $C_c^\infty(G_T \rtimes \mathbb{R})$.

Proof. Let ξ^0, \dots, ξ^q be elements of $\mathcal{A} \rtimes_\sigma \mathbb{R}$, $f^j \in C^\infty(G \rtimes_\delta \mathbb{R})$ the Fourier transform of ξ^j for $0 \leq j \leq q$. In terms of f^0 and the partial derivatives of f^j , $\hat{\phi}$ is expressed as the following q -cocycle

$$\sum_{j \in \mathfrak{S}_q} \int_{\lambda_0 + \dots + \lambda_q = 0} f^0[\lambda_0] \frac{\partial}{\partial x_{j_1}} \hat{f}^1[\lambda_1] \dots \frac{\partial}{\partial x_{j_q}} \hat{f}^q[\lambda_q] dx_{j_1} \dots dx_{j_q}.$$

For each $j \in \mathfrak{S}_q$, the corresponding term

$$\int_{\lambda_0 + \dots + \lambda_q = 0} f^0[\lambda_0] \frac{\partial}{\partial x_{j_1}} \hat{f}^1[\lambda_1] \dots \frac{\partial}{\partial x_{j_q}} \hat{f}^q[\lambda_q] dx_{j_1} \dots dx_{j_q}$$

is the convolution product at $\lambda = 0$ of the Fourier transforms of f^0 and $\partial_{j_k} f^k$ ($1 \leq k \leq q$). Thus it is equal to the Fourier transform at $\lambda = 0$ of their products, namely,

$$\int dt f^0(\gamma_0) \frac{\partial f^1(\gamma_1)}{\partial x_{j_1}} \dots \frac{\partial f^q(\gamma_q)}{\partial x_{j_q}} dx_1 \dots dx_q.$$

Combining all these terms for $j \in \mathfrak{S}_q$, we obtain the assertion. \square

Definition 11. (c.f. [5]) A cyclic k -cocycle ψ over a pre-C*-algebra \mathcal{A} is said to be *anabelian* when the $(k + 1)$ -linear map

$$\psi_g : (f^0, \dots, f^k) \mapsto \psi(gf^0, \dots, f^k)$$

is again a cyclic k -cocycle on \mathcal{A} for any element g in the center $Z(M(\mathcal{A}))$ of the multiplier of \mathcal{A} .

Lemma 12. *In order for a cyclic q -cocycle ψ to be anabelian, it is necessary and sufficient to have $\psi(g, f^1, \dots, f^q) = 0$ for any $g \in Z(M(\mathcal{A}))$ and $f^j \in \mathcal{A}$ for $1 \leq j \leq q$.*

Proof. First let us show that the assumption of the lemma implies the following equality for any element $g \in Z(M(\mathcal{A}))$:

$$(2) \quad \psi(gf^0, f^1, \dots, f^q) = \psi(f^0, gf^1, \dots, f^q).$$

By the cyclic cocycle condition on ψ , this is equivalent to $\psi(g, f^1, \dots, f^q) = 0$ for any $g \in Z(M(\mathcal{A}))$, $f^j \in \mathcal{A}$. Indeed, when this is satisfied, by the cyclicity condition we have

$$\psi(f^j, \dots, f^q, g, f^0, \dots, f^{j-1}) = 0$$

for any j . Together with the cocycle condition

$$\begin{aligned} 0 &= b\psi(f^0, g, f^1, \dots, f^q) = \psi(f^0 g, f^1, \dots, f^q) - \psi(f^0, g f^1, f^2, \dots, f^q) \\ &\quad + \sum_{j=2}^q (-1)^j \psi(f^0, g, \dots, f^{j-1} f^j, f^{j+1}, \dots, f^q) + (-1)^q \psi(f^q f^0, g, f^1, \dots, f^{q-1}) \end{aligned}$$

implies the equality (3).

The cocycle property of ψ_g follows from that of ψ and the fact that g commutes with any element of \mathcal{A} . On the other hand, the cyclicity property is an immediate consequence of the equality (3). \square

Definition 13. Let G_T^u denote the kernel $\{\gamma \in G_T \mid \delta(\gamma) = 1\}$ of the groupoid homomorphism $\delta : G_T \rightarrow \mathbb{R}$. We call G_T^u the unimodular part of G_T .

Proposition 14. Suppose that the unimodular part G_T^u acts ergodically on T . Then the cocycle ψ is anabelian.

Proof. Let g be an element of the center of $M(C_c^\infty(G_T \rtimes \mathbb{R}))$. It is represented by an G -invariant L^∞ function on $T \times \mathbb{R}$. By ergodicity assumption the “restriction” of this function to $T \times \{t\}$ is some constant $g(t)$ for each $t \in \mathbb{R}$. Then we can express $\psi(g, f^1, \dots, f^q)$ as

$$\psi(g, f^1, \dots, f^q) = \int_t dt \int_{T \times \{t\}} g(t) df^1 \cdots df^q.$$

For each $t \in \mathbb{R}$, the integral $g(t) \int_{\theta_t \bar{T}} df^1 \cdots df^q$ of a closed form is equal to 0. \square

Thus for each g in the center of $W(M; F) \rtimes_\sigma \mathbb{R}$, we obtain a cyclic q -cocycle

$$\psi_g : (f^0, \dots, f^q) \mapsto \psi(g f^0, \dots, f^q).$$

Remark 15. The q -cocycle ψ_g over $C_c^\infty(G_T \rtimes \mathbb{R})$ is given by the invariant 1-form $g(t)dt$. By [5], ψ_g is a q -trace for any g .

Consequently for each $u \in K_1(C^*(M; F) \rtimes_\sigma \mathbb{R})$ we obtain a linear map $g \mapsto \langle \psi_g, u \rangle$ over $Z(W(M; F_u))$. This measure over the flow of weights is invariant under θ because of the θ -invariance of ψ . Note that when G_T^u acts ergodically on T , the center of $W(M; F)$ is trivial as its elements are represented by G_T -invariant functions on T . To summarize, we have proved the following proposition:

Proposition 16. Let (M, F) be a transversely affine foliation whose unimodular part is ergodic. When the dual class $\psi = \hat{\phi}$ of the transverse fundamental class pairs nontrivially with $K_q(C^*(G_T \rtimes \mathbb{R}))$, the factor $W(M, F)$ is of type III.

2.3. Geometric cohomology class corresponding to the dual fundamental class. Let ϕ be a σ -invariant cyclic q -cocycle on \mathcal{A} . Then, we get a new cyclic $(q+1)$ -cocycle $i_D \phi$ determined by

$$i_D \phi(f^0 df^1 \cdots df^{q+1}) = \sum_{1 \leq j \leq q+1} (-1)^j \phi(f^0 df^1 \cdots df^{j-1} D f^j df^{j+1} \cdots df^{q+1}).$$

By [4] [7], $\phi \mapsto i_D \phi$ is compatible with the Connes-Thom isomorphism $\Psi : K_*(\mathcal{A}) \rightarrow K_{*+1}(\mathcal{A} \rtimes_\sigma \mathbb{R})$ in the K -theory. Together with the Morita equivalence $\mathcal{A} \rtimes_\sigma \mathbb{R} \simeq C^*(M; F_u)$, for any $x \in K_{q+1}(\mathcal{A})$ one has

$$\langle i_D \phi, x \rangle = \langle \hat{\phi}, \Psi(x) \rangle.$$

Now we are going to seek an geometric representation of $i_D\phi$. To this end, we employ the transformation Φ of [6] from the periodic cyclic cohomology $HP^j(C^\infty(M; F))$ to the usual cohomology $\oplus_k H^{j+q+2k}(M)$ with complex coefficients, which is given as the composition of the natural projection $HP^*(\mathcal{A}_M) \rightarrow H_\tau^*(BG)$ of the cyclic cohomology to the cohomology of the classifying space of G and the pullback map $H_\tau^*(BG) \rightarrow H_\tau^*(M)$ induced by the classifying map $M \rightarrow BG$.

This operation allows us to check if the cyclic $(q+1)$ -cocycle $i_D\phi$ is nontrivial. Specifically, we have the assembly map $\mu : K_{\tau,*}^0(BG) \rightarrow K_0(C^*(M; F))$ that is compatible with Φ and the Chern character $ch_{\tau,*} : K_{\tau,*}^0(BG) \rightarrow H_*(M)$ in homology:

$$\langle i_D\phi, \mu x \rangle = \langle \Phi(i_D\phi), ch_{\tau,*}x \rangle.$$

Thus the cyclic cocycle $\hat{\phi}$ pairs nontrivially with $K_q(\mathcal{A} \rtimes \mathbb{R})$ when the cohomology class $\Phi(i_D\phi)$ is not zero.

The localization of the periodic cyclic cohomology over open sets [6] allows us to investigate an explicit cocycle corresponding to $\Phi(i_D\phi)$. For each open set U of M , put $\mathcal{A}_U = C_c^\infty(U; F|_U)$. The assignment of the periodic cyclic cohomology group $HP^*(\mathcal{A}_U)$ to each open set U defines a sheaf on M .

Remark 17. Recall that, as in Remark 9, we have a manifold $\wedge^q \tau^*$ of dimension $n+1$ endowed with a foliation F_u of codimension $q+1$. This foliation, combined with the action of the structure group \mathbb{R}_+^\times on the fibers of the projection $\wedge^q \tau^* \rightarrow M$ determines a foliation F' of codimension q on $\wedge^q \tau^*$. The algebra $C_c^\infty(\wedge^q \tau^*; F')$ is Morita equivalent to $C_c^\infty(M; F)$.

Let (x_1, \dots, x_q, y) denote the standard coordinate of $\mathbb{R}^q \times \mathbb{R}_+^\times$. Thus x_j runs through \mathbb{R} for $1 \leq j \leq q$ while y runs through \mathbb{R}_+^\times . The choice of a covering $(U_i)_i$ of M by foliation charts on T determines a groupoid homomorphism g of $\coprod_{i,j} G_{U_i}^{U_j}$ into the diffeomorphism group $\text{Diff}(\mathbb{R}^q)$ of \mathbb{R}^q . We identify $\mathbb{R}^q \times \mathbb{R}_+^\times$ with the total space of the positive density bundle $(\wedge^q T^* \mathbb{R}^q)_+$. Thus the group $\text{Diff}(\mathbb{R}^q)$ acts on $\mathbb{R}^q \times \mathbb{R}_+^\times$ by

$$g.(x_1, \dots, x_q, y) = (g(x_1, \dots, x_q), y|dg_{(x_1, \dots, x_q)}|)$$

The differential forms (or, any other geometric objects) on $\mathbb{R}^q \times \mathbb{R}_+^\times$ which are invariant under g can be pulled back to $\wedge^q \tau^*$.

When the foliation is transversely affine, the cocycle g can be reduced to the affine transformation group. Let dt denote the closed 1-form $\frac{dy}{y}$ on $\mathbb{R}^q \times \mathbb{R}_+^\times$. This is invariant under g , and defines a closed 1-form again denoted by dt on $\wedge^q \tau^*$.

Definition 18. An open set U in M is said to be *straight* when we have a complete transversal T in U with trivial holonomy for $F|_U$.

Remark 19. As any foliation chart is always straight, M admits an open covering by straight open sets.

Let U be a straight open set. The choice of the complete transversal T in U determines a “ t -coordinate” t on the leaves of $F|_U$ such that T becomes the set of the points whose t_i coordinate is equal to 0. This gives a Morita equivalence $\mathcal{A}_U \simeq C_c(T) \otimes \mathcal{K}(\mathbb{R})$, where $\mathcal{K}(\mathbb{R})$ denotes the algebra of smooth functions in two real variables endowed with the convolution product.

Let η denote the cyclic 1-cocycle on $\mathcal{K}(\mathbb{R})$ defined by

$$\eta(f^0, f^1) = \int_{\mathbb{R}^2} f^0(t, t') f^1(t', t) (t - t') dt dt'.$$

Lemma 20. *Let U be a straight open set in M with complete transversal S , and $[G_{U,S}]$ the transverse fundamental class of the restriction algebra \mathcal{A}_U . Then the $(q+1)$ -cocycle $i_D[G_{U,S}]$ on \mathcal{A}_U is equivalent to the cup product $[S] \# \eta$ on $C_c^\infty(S) \otimes \mathcal{K}(\mathbb{R})$.*

Proof. The Morita equivalence between \mathcal{A}_U and $C_c^\infty(S) \otimes \mathcal{K}(\mathbb{R})$ reduces to the equivalence of the holonomy relation on $U \simeq S \times \mathbb{R}$ and the relation on $S \times \mathbb{R}$ whose equivalence classes are of the form $\{s\} \times \mathbb{R}$ for $s \in S$.

Let γ be an $F|_U$ holonomy on U of the source x the range y . Then we can consider the t -coordinate t_x, t_y of x and y . On the other hand we have a unique point s in S that is on the same $F|_U$ -leaf as x and y . Then γ corresponds to $(s, t_x, t_y) \in s \in S \times \mathbb{R} \times \mathbb{R}$. The difference $t_y - t_x$ of the second component (“ t -coordinate”) of the endpoints of a holonomy γ is exactly equal to $\log \delta \gamma$. \square

Proposition 21. *With the notation as above, we have $\Phi(i_D \phi) = dt$ in the non-compact support de Rham cohomology.*

Proof. We use two triple complexes of [6] which relate the periodic cyclic cohomology $HP^*(\mathcal{A}_M)$ to the Čech cohomology $\check{H}(M; \mathbb{C})$. Let $(U_i)_{i \in I}$ be a covering of M by straight open sets. We suppose there is a total ordering on the index set I . First, we have a triple complex

$$\Gamma^{a,b,c} = \prod_{i_1 < \dots < i_c} \Omega_F^{b-a}(U_{i_1} \cap \dots \cap U_{i_c})$$

where $\Omega_F^k(U)$ denotes the \mathcal{A}_U -bimodule $(\mathcal{A}_U^{\otimes k} \oplus \mathcal{A}_U^{\otimes(k+1)})'$. This triple complex gives a resolution of the (B, b) -bicomplex of \mathcal{A}_M .

On the other hand we have another triple complex

$$\Gamma'^{a,b,c} = \prod_{i_1 < \dots < i_c} \Omega_{b-a}^\tau(U_{i_1} \cap \dots \cap U_{i_c})$$

where $\Omega_k^\tau(U)$ denotes the space of F_U -holonomy invariant transverse currents on U . We have a map $\Psi : \Omega_k^\tau(U) \rightarrow \Omega_F^k(U)$ for each k and U . Note that under the linear holonomy assumption we do not need to handle the matrix algebra $\mathcal{A}_U \otimes M_2(\mathbb{C})$.

For each $i \in I$, let T_i be a transversal in U_i . The $(q+1)$ -cocycle $i_D \phi$ is represented by the family

$$(i_D[G_{T_i}])_{i \in I} \in \prod_{i \in I} \Omega_F^2(U_i).$$

By Lemma 20, under the identification $\mathcal{A}_{U_i} \simeq C_c^\infty(T_i) \otimes \mathcal{K}(\mathbb{R})$, $i_D[G_{T_i}]$ corresponds to the cup product $[T_i] \# \eta$.

On the other hand, η represents the trivial class in the first cyclic cohomology group $HC^1(\mathcal{K}(\mathbb{R}))$. It can be written as $B\eta_0$ where η_0 is defined by $\eta_0(f) = \int_{\mathbb{R}} f(t, t) dt$. Let ϕ be the transverse fundamental current. Then $i_D \phi$ is cohomologous to the Čech 1-cocycle $(\phi \# (t_i - t_j)_{ij}) \in (\Omega_F^1(U_{ij}))$.

Thus we are led to the 1-cocycle $((t_i - t_j)\phi)_{ij} \in \prod_{ij} \Omega_1^\tau(U_{ij})$ as another representation of $i_D \phi$. This lies actually in the subspace $\prod_{ij} H_\tau^1 U_{ij}$. It represents a 1-cocycle in the Čech complex of the orientation sheaf of τ . This corresponds to

the 1-cocycle $(t_i - t_j)_{ij}$ in the Čech complex of the constant sheaf of \mathbb{C} , which is equivalent to the de Rham cohomology class represented by the 1-form dt . \square

Theorem 22. *When the cohomology class $dt \in H^1(M)$ is nontrivial, the von Neumann algebra $W(M; F)$ is of type III.*

Note that the cohomology class of the 1-form dt is independent of the choice of the transversal T .

2.4. Determination of λ in the case of type III_λ foliation algebra. When the algebra $W(M; F)$ is of type III_λ , the pairing of the $(q+1)$ -cocycle $i_D\phi$ with the K -group retains the value of λ .

Definition 23. The submanifold

$$\tilde{T} = \{r\gamma \mid \gamma \in G^u, s\gamma \in T\}$$

of M is called the *unimodular span* of T .

The unimodular span \tilde{T} is a submanifold in M of codimension 1 which is transverse to the flow θ . Let X_θ be the vector field on M that generates the flow θ , dt the 1-form on M characterized by $\langle dt, X_\theta \rangle = 1$ and $\langle dt, X \rangle = 0$ for the vectors tangent to \tilde{T} .

Proposition 24. *The subgroup $\langle \psi, K_q(C^*(G_T \rtimes_\sigma \mathbb{R})) \rangle = \langle i_D\phi, K_{q+1}(C^*(M; F)) \rangle$ of \mathbb{R} contains $\log(\lambda)$.*

Proof. Let N be the unimodular span of the transversal T . The flow θ_λ determines a self map f of N . This map preserves the fundamental class of N while M is equivalent to the mapping torus M_f of N . There is a compact support cohomology class $c \in H^{n-1}(M)$ which is an integer coefficient cohomology class. The cohomology class $\text{Td}(\tau_{\mathbb{C}}) \cup c$ can be expressed as $ch_{\tau,*}(x)$ for some $x \in K^0(F^*)$. Since the constant term of the Todd class $\text{Td}(\tau_{\mathbb{C}})$ is 1, we have $\langle i_D\phi, \mu(x) \rangle = \langle dt, c \cap [M] \rangle = -\log \lambda$. \square

Remark 25. Let $K(M; F)$ be the K -set of Moriyoshi [8], which is a geometrically defined subgroup of \mathbb{R} . Proposition 24 establishes the conjectured equivalence

$$\log(S(W^*(M; F)) \setminus \{0\}) = K(M; F)$$

of the S -set $S(W^*(M; F))$ of the foliation von Neumann algebra when F is transversely affine.

See [8] for the notations ω , k_ω , ρ_ω and H . The F -basic differential forms of [8] are exactly the holonomy invariant transverse differential forms. The existence of a transverse coordinate with respect to which the holonomy maps becomes affine implies the existence of F -basic q -form ω on M . The group cocycle $\log \rho_\omega$ corresponds to the time derivative of the transverse fundamental class under the modular automorphism. In fact it is enough to have a transversal T with a coordinate system (x_1, \dots, x_q) such that the action of the reduced groupoid G_T becomes affine. Then the volume form on T with respect to the coordinate system x becomes projectively invariant under G_T . Note that in the proof of Proposition 5 only the projective invariance of the transverse density is utilized to show the invariance of the transverse fundamental class under modular automorphism group.

The first cohomology class $k_\omega = [\log \rho_\omega] \in H^1(M)$ is equal to the class of the 1-form *spectrum* in Proposition 21. On the other hand, the wedge products of the

holonomy invariant transverse 1-forms with the cohomology class c in the proof of Proposition 24 are all 0, since the tangent space of the unimodular span N and F_x generate $T_x M$ at each point $x \in N$.

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